**Math 10B with Professor Stankova**
**Worksheet, Discussion #24; Thursday, 4/19/2018**
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# 1 Gaussian Elimination

## 1.1 Concepts

1. In order to solve a system of equations to find the solution or determine if there are zero or infinitely many solutions, use Gaussian elimination on the **augmented matrix**, a matrix formed by appending the answer vector to the original matrix. A system of equations is **consistent** if there is at least one solution and **inconsistent** if there are no solutions.

## 1.2 Example

2. Use Gaussian elimination on the following augmented matrix. Write the equations these correspond to.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & 2 \\ -3 & 0 & -2 & -1 \end{array}\right)$$

**Solution:** Add 3 times the first row to the third row to get

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & 2 \\ -3 & 0 & -2 & -1 \end{array}\right) \xrightarrow{III+3I} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & 2 \\ 0 & 6 & 1 & 8 \end{array}\right) \xrightarrow{I+2II, III+6II} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 7 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & -5 & 20 \end{array}\right)$$

$$\xrightarrow{III/-5} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 7 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right) \xrightarrow{I+III, II+III} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -4 \end{array}\right) \xrightarrow{-II} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right)$$

Therefore the solution is $(3, 2, -4)$.

## 1.3 Problems

3. True **FALSE** As soon as we see a row like $(000\ldots0|0)$ during Gaussian elimination, we know that the system will have infinitely many solutions.

**Solution:** See Problem 7.

4. **TRUE** False If we see a row like $(000\dots0|0)$ then we know the determinant of the matrix.

**Solution:** The determinant will have to be 0 because there will be 0 or $\infty$ solutions.

5. Come up with an example of a consistent system of equations with 3 equations and 2 variables. Give an example of an inconsistent system of linear equations with 2 equations and 3 variables.

**Solution:** One example for a consistent system is $x+y=11, 2x+3y=23, -x+2y=-8$ with solution $x=10, y=1$.

One example for an inconsistent system is $x+2y+3z=2, x+2y+3z=1$, because it has no solutions.

6. Find conditions on $a, b$ such that the following system has no solutions, infinitely many, and a unique solution.

$$\begin{cases} x+ay=2 \\ 4x+8y=b \end{cases}$$

**Solution:** We want to solve the equation

$$\begin{pmatrix} 1 & a \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ b \end{pmatrix}.$$

We know that this has a unique solution if the determinant is nonzero so we need $8-4a \neq 0$ or $a \neq 2$. For all $a \neq 2$ and any $b$, this has a unique solution.

Now if $a=2$, we know that this solution has zero or infinite solutions. To tell, we need to use Gaussian elimination. Putting it into an augmented matrix and solving gives us

$$\left(\begin{array}{cc|c} 1 & a & 2 \\ 4 & 8 & b \end{array}\right) \xrightarrow{II-4I} \left(\begin{array}{cc|c} 1 & a & 2 \\ 0 & 0 & b-8 \end{array}\right)$$

Thus if $b \neq 8$, then we have an inconsistent system and the system has no solutions. If $b=8$, then there are infinitely many solutions.

7. Use Gaussian elimination to solve the following system of equations:

$$\begin{cases} 2x_1 + x_2 - x_3 = 4 \\ -4x_1 - 2x_2 + 2x_3 = -6 \\ 6x_1 + 3x_2 - 3x_3 = 12 \end{cases}$$

**Solution:** Writing this as an augmented matrix, we get

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ -4 & -2 & 2 & -6 \\ 6 & 3 & -3 & 12 \end{array}\right) \xrightarrow{II+2I, III-3I} = \left(\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So this system of equations has no solutions.

8. Find $\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 2 & 5 & 2 \end{pmatrix}^{-1}$.

**Solution:** We need to use Gaussian elimination to reduce

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 5 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{III-2I} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{I-3II, III+II} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -3 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right) \xrightarrow{I+2III, II-III} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & 2 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right)$$

Thus the inverse is $\begin{pmatrix} -3 & -1 & 2 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix}$.

9. Use Gaussian elimination to solve the following system of equations:

$$\begin{cases} x_1 - 2x_2 - 6x_3 = 5 \\ 2x_1 + 4x_2 + 12x_3 = -6 \\ x_1 - 4x_2 - 12x_3 = 9 \end{cases}$$

**Solution:** Using Gaussian elimination gives

$$\left(\begin{array}{ccc|c} 1 & -2 & -6 & 5 \\ 2 & 4 & 12 & -6 \\ 1 & -4 & -12 & 9 \end{array}\right) \xrightarrow{II-2I,III-I} \left(\begin{array}{ccc|c} 1 & -2 & -6 & 5 \\ 0 & 8 & 24 & -16 \\ 0 & -2 & -6 & 4 \end{array}\right)$$

$$\xrightarrow{II/8,III/-2} \left(\begin{array}{ccc|c} 1 & -2 & -6 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \end{array}\right) \xrightarrow{I+2II,III-II} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solving, we get $x_1 = 1, x_2 = -2 - 3x_3$ and $x_3$ can be whatever it wants. This means there are an infinite number of solutions, that the determinant of the original matrix is 0, and the general solution is of the form $\begin{pmatrix} 1 \\ -2-3x_3 \\ x_3 \end{pmatrix}$.

10. Use Gaussian elimination to solve the following system of equations:

$$\begin{cases} z - 3y = -6 \\ x - 2y - 2z = -14 \\ 4y - x - 3z = 5 \end{cases}$$

**Solution:** First write it in an augmented matrix.

$$\left(\begin{array}{ccc|c} 0 & -3 & 1 & -6 \\ 1 & -2 & -2 & -14 \\ -1 & 4 & -3 & 5 \end{array}\right) \xrightarrow{I\leftrightarrow II} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -14 \\ 0 & -3 & 1 & -6 \\ -1 & 4 & -3 & 5 \end{array}\right) \xrightarrow{III+I} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -14 \\ 0 & -3 & 1 & -6 \\ 0 & 2 & -5 & -9 \end{array}\right)$$

$$\xrightarrow{I-2/3II,III+2/3II} \left(\begin{array}{ccc|c} 1 & 0 & -8/3 & -10 \\ 0 & -3 & 1 & -6 \\ 0 & 0 & -13/3 & -13 \end{array}\right) \xrightarrow{III*3/-13} \left(\begin{array}{ccc|c} 1 & 0 & -8/3 & -10 \\ 0 & -3 & 1 & -6 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{I+8/3III,II-III} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{II/-3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Thus the solution is $(-2, 3, 3)$.

### 1.4 Extra Problems

11. Use Gaussian elimination to solve the following system of equations:

$$\begin{cases} z - 3y = -2 \\ 3y - 4x - 3z = 2 \\ 2z - x - y = -5 \end{cases}$$

**Solution:** Solving gives the solution $(x, y, z) = (1, 0, -2)$.

12. Use Gaussian elimination to solve the following system of equations:

$$\begin{cases} 2x + 4y - 4z = 0 \\ 5x + y = 6 \\ x - 7y + 8z = -6 \end{cases}$$

**Solution:** Using Gaussian elimination gives

$$\left(\begin{array}{ccc|c} 1 & 0 & 2/9 & 0 \\ 0 & 1 & -10/9 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Thus, there are no solutions.

13. Find $\begin{pmatrix} 1 & 4 & 3 \\ 1 & 1 & 1 \\ 3 & -1 & 0 \end{pmatrix}^{-1}$.

**Solution:** Use Gaussian elimination to get the inverse is $\begin{pmatrix} 1 & -3 & 1 \\ 3 & -9 & 2 \\ -4 & 13 & -3 \end{pmatrix}$.

## 2 Eigenvalues and Eigenvectors

### 2.1 Concepts

14. An eigenvalue eigenvector pair for a square matrix $A$ is a scalar $\lambda$ and nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. To find this, we write $\lambda\vec{v} = \lambda I\vec{v}$ and bring this to the other side to

get $(A - \lambda I)\vec{v} = 0$. Since $\vec{v}$ is nonzero, this means that $(A - \lambda I)\vec{w} = 0$ has at least two solutions (since the trivial solution is a solution), and hence there must be an infinite number of solutions and $\det(A - \lambda I) = 0$.

So to find the eigenvalues, we solve $\det(A - \lambda I) = 0$. For a particular eigenvalue, to find the associated eigenvector, we have to use Gaussian elimination on $A - \lambda I$ to get the general solution.

## 2.2 Example

15. Find the eigenvalue and associated eigenvectors of $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$.

**Solution:** We take the determinant of $A - \lambda I = \begin{pmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{pmatrix}$ which is $(1-\lambda)(2-\lambda) - 0 = (\lambda - 1)(\lambda - 2)$. Therefore the eigenvalues are 1 and 2.

The associated eigenvector of 1 is gotten by looking at $A - 1I = A - I = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and we want to solve $(A - I)\vec{v} = \vec{0}$ and one nontrivial solution is $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which is an eigenvector.

For $\lambda = 2$, we look at $A - 2I = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$ and a nontrivial eigenvector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

## 2.3 Problems

16. **TRUE** False Associated to every eigenvalue is an eigenvector and vice versa

17. **TRUE** False If 2 is an eigenvalue for $A$, then 4 is an eigenvalue for $A^2$.

**Solution:** Let $\vec{v}$ be the associated eigenvector so that $A\vec{v} = 2\vec{v}$. Then $A^2\vec{v} = A(A\vec{v}) = A(2\vec{v}) = 2^2\vec{v} = 4$ so $A^2$ has an eigenvalue 4 with eigenvector $\vec{v}$.

18. **TRUE** False If $\det(A) = 0$, then 0 has to be an eigenvalue of $A$.

**Solution:** This satisfies $\det(A - 0I) = \det(A) = 0$.

19. True **FALSE** If 2 is an eigenvalue of $A$ and 3 is an eigenvalue of $B$, then $2 \cdot 3 = 6$ is an eigenvalue of $AB$.

20. True **FALSE** For each eigenvalue, there is only one choice of eigenvector.

**Solution:** We can choose any multiple of an eigenvector.

21. Find the eigenvalues and eigenvectors of $\begin{pmatrix} 1 & 3 \\ 9 & -5 \end{pmatrix}$.

**Solution:** We have to look at $\det(A - \lambda I) = (1-\lambda)(-5-\lambda) - 27 = \lambda^2 + 4\lambda - 32 = (\lambda+8)(\lambda-4)$. So the eigenvalues are $\lambda = 4, -8$. For the eigenvalue 4, an eigenvector is gotten by looking at $A - 4I = \begin{pmatrix} -3 & 3 \\ 9 & -9 \end{pmatrix}$ and an eigenvector is $\begin{pmatrix} 1 & 1 \end{pmatrix}$. For the eigenvalue $-8$, an eigenvector is gotten by looking at $A + 8I = \begin{pmatrix} 9 & 3 \\ 9 & 3 \end{pmatrix}$ and an eigenvector is $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

22. Find the eigenvalues and eigenvectors of $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

**Solution:** We look at $\det(A - \lambda I) = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda$ so $\lambda = 0, 2$. For $\lambda = 0$, an eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $\lambda = 2$, we look at $A - 2I = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$ and get an eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

23. Find the eigenvalues of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

**Solution:** We look at $\det(A - \lambda I) = (1-\lambda)^2 = 0$ so $\lambda = 1$ is the only eigenvalue.

24. Construct a matrix with eigenvalues 3 and $-1$.

**Solution:** We can just take $\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.

25. Find the eigenvalues of $\begin{pmatrix} 2 & 4 & 4 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}$.

**Solution:** We take the determinant of $A - \lambda I = \begin{pmatrix} 2-\lambda & 4 & 4 \\ -1 & -\lambda & -1 \\ 1 & 0 & 1-\lambda \end{pmatrix}$ which is $-\lambda(\lambda - 1)(\lambda - 2)$ so the eigenvalues are $\lambda = 0, 1, 2$.